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Dynamic Response of a Long Case-Bonded Viscoelastic Cylinder

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Time-dependent pressures are applied in an encased viscoelastic cylinder and on the surface of the case. The resulting dynamic response of the cylinder-case system is the subject of a theoretical analysis. The viscoelastic material of the cylinder is assumed to be incompressible and a forced vibration is therefore excited without initial wave effects. The cylinder is viscoelastic in shear, showing short-time elastic behavior and delayed elasticity. The displacement, the circumferential stress, and the radial stress are investigated. If a time-dependent pressure is applied to the case, the radial bond stress at the cylinder-case interface is periodic and shows tensile peaks for high values ($\sim 10^4$) of the ratio of Young's modulus of the shell to the rubbery shear modulus of the cylinder. The stresses are damped exponentially to the compressive quasi-static solution. Analytical solutions are presented for step loading and standard linear viscoelastic shear behavior. A numerical procedure using measured values of the relaxation function is indicated. The solutions are relevant to a compressible cylinder for times long compared to the passage-time of a dilatational wave.

Introduction

DESIGN considerations for solid propellant rocket motors have led to detailed investigations of a structural system consisting of a hollow viscoelastic cylinder which is contained in a thin elastic case (shell). A number of problems have been formulated and solved for the axially symmetric plane-strain configuration. Quasi-static linear viscoelasticity theory has generally been used.

The quasi-static response of a cylinder-shell system to a time-dependent internal pressure is discussed by Bland.[1] Williams et al.[2] have also surveyed several problems of this type. A short review of the cylinder problem for quasi-static viscoelasticity is included in a recent paper by Rogers and Lee.[3]

Until quite recently, dynamic effects were not taken into account in determining the viscoelastic cylinder response. Emphasis was placed rather on improving the material specification and on including the effect of an ablating inner surface. Increasing importance of the vibrations of solid propellant rocket motors has, however, necessitated research in the area of vibrating viscoelastic cylinders.

In the present paper a long encased cylinder is subjected to time-dependent pressures at the inner and outer surfaces, inertia being taken into account. The displacements, the circumferential stresses, and the radial stresses are investigated. Of particular interest is the radial stress at the interface between the cylinder and the shell. The influence of structural and material parameters on the radial bond stress has been given special attention.

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The sudden application of a constant pressure in a cylindrical cavity generates a compressional wave. The propagation of such a wave in an infinite medium has been investigated by Selberg.[4] If a time-dependent pressure is applied in an encased cylinder, the compressional wave interacts with the case and the circular inner boundary. Since the dilatational wave velocity is very high, it is to be expected that a steady-state forced vibration is established in the cylinder-case system within a very short time.

In this paper, the viscoelastic material of the cylinder is assumed to be incompressible. For dynamic problems the assumption of incompressibility implies an infinite dilatational wave velocity. The application of pressure in an incompressible viscoelastic cylinder therefore initiates a damped vibration without initial wave effects.

The assumption of incompressibility is based on observed mechanical behavior of solid propellant binders. These rubbery materials are often considered as essentially incompressible. No real material can, however, be completely incompressible. To appraise the value of the present solutions we have to contrast them with quasi-static solutions, and with the transient solution that was described earlier. For a compressible cylinder the validity of quasi-static solutions requires small changes of stress to occur in times much larger than the wave propagation time across the body. Thus, the solutions for a suddenly applied pressure must be interpreted as having meaning only for such a ramp loading, and they should not be interpreted strictly as the response to a mathematical step function. It is evident therefore that, for a suddenly applied pressure and for the geometry which is considered in this paper, a dynamic solution under the assumption of incompressibility is superior to a quasi-static solution. The possibly important initial wave effects which occur in a compressible cylinder are not included, however, and the present solution

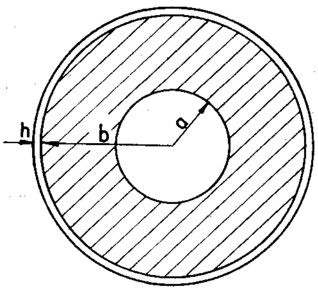

Fig. 1 Viscoelastic cylinder contained in a thin elastic shell.

can therefore only be relevant to times that are long compared with the passage of a dilatational wave. For a material with a high bulk modulus, such as a solid propellant binder, the passage time of a dilatational wave is small. There is thus a significant time range in which the present solutions improve on a quasi-static analysis.

If a pressure is suddenly applied at the inner surface, the radial stresses at the cylinder-case interface are periodic and they are always compressive. If a pressure is applied to the shell, the radial bond stresses can show tensile peaks for certain values of the material parameters. The bond stresses are always compressive for the quasi-static response.

The present solutions may have application to the response of solid propellant grains during the first moments of launching a missile from a silo. At ignition only normal pressures at the inner surface of the viscoelastic cylinder occur; however, as the exhaust gases fill the silo, normal pressures are also applied to the shell. Stages which are not operative at launching are subjected to an external pressure.

Solid propellant binders belong to a class of viscoelastic materials which show an initial elastic shear response and delayed shear elasticity. Of great importance in the present problem is the ratio of Young's modulus of the elastic shell to the rubbery shear modulus of the binder. For high values of this ratio the previously mentioned tensile bond stresses occur. The viscoelastic cylinder, however, dampens the periodic stresses, and the quasi-static compressive stress is reached.

To exhibit the type of response which can be expected for viscoelastic cylinders of the class of materials under consideration, analytical solutions have been derived for a cylinder of standard linear viscoelastic solid. The governing equations have also been cast in a form which allows a solution by numerical methods.

Encased Viscoelastic Cylinder

We consider a long viscoelastic cylinder (Fig. 1) with a circular port of constant radius a and an outer radius b at which the cylinder is bonded to a thin elastic shell. The cylinder-shell system is subjected to time-dependent but spatially uniform normal pressures. Under the conditions of axial symmetry and plane strain, the only nontrivial equation of motion for the cylinder is

$$\partial\sigma_r/\partial r + (\sigma_r - \sigma_\theta)/r = \rho_p(\partial^2 u/\partial t^2) \tag{1}$$

With the strain-displacement relations $\epsilon_r = \partial u/\partial r$, $\epsilon_\theta = u/r$, and $\epsilon_z = 0$, the assumption of incompressibility can be expressed in the form

$$\partial u/\partial r + u/r = 0 \tag{2}$$

The solution of Eq. (2) is simply

$$u = k(t)/r \tag{3}$$

where $k(t)$ is a function of t only.

Let s_{ij} and e_{ij}, respectively, denote the components of deviatoric stress and strain. The stress-strain relation for the incompressible viscoelastic material can then be expressed in the form

$$s_{ij}(r, t) = 2\int_{0-}^{t} G(t - s)de_{ij}(r, s)ds \tag{4}$$

In Eq. (4), $G(t)$ is the relaxation function in shear. From Eqs. (4) and (3) we derive

$$\sigma_r - \sigma_\theta = 2\int_{0-}^{t} G(t - s)d(\epsilon_r - \epsilon_\theta) = \left(\frac{-4}{r^2}\right)\int_{0-}^{t} G(t - s)dk(s) \tag{5}$$

The expressions for $u(r, t)$ and $\sigma_r - \sigma_\theta$, respectively, Eqs. (3) and (5), are substituted into the equation of motion (1). The resulting equation for σ_r is subsequently integrated with respect to r:

$$\sigma_r = \rho_p\ddot{k}(t)\ln(r) - \left(\frac{2}{r^2}\right)\int_{0-}^{t} G(t - s)dk(s) + A(t) \tag{6}$$

The functions $A(t)$ and $k(t)$ are determined by taking into account the initial conditions and the boundary conditions at $r = a$ and at $r = b$.

It is assumed that, prior to $t = 0$, the system is at rest. Thus

$$k(t) = \dot{k}(t) \equiv 0 \qquad \text{for } t < 0 \tag{7}$$

The boundary condition at $r = b$ is derived from the equation of motion of an element of the thin surrounding shell:

$$-\sigma_r]_{r=b} - \sigma_e - \sigma_c(h/b) = \rho_c h\ddot{u}_r]_{r=b} \tag{8}$$

In Eq. (8), $\sigma_r]_{r=b}$, σ_c, and σ_e are, respectively, the radial stress in the cylinder at $r = b$, the circumferential stress in the case, and the normal pressure that is applied at the outside of the case. The mass density of the case is denoted by ρ_c. Using the relation between the circumferential stress and the circumferential strain in the case, and the requirement that the circumferential strain is continuous at the cylinder-case interface we obtain

$$\sigma_r]_{r=b} = -\sigma_e - (h/b)(E'/b^2)k(t) - \rho_c(h/b)\ddot{k}(t) \tag{9}$$

where $E' = E_c/(1 - \nu_c^2)$ and ν_c is Poisson's ratio for the case. Solutions are determined for the following two sets of boundary conditions.

Problem I: a time-dependent pressure is applied inside the cylinder; no pressure on the outside of the case:

$$\begin{aligned} &\text{at } r = a \qquad \sigma_r = -\sigma_i(t) \\ &\text{at } r = b \qquad \sigma_r = -(h/b)(E'/b^2)k(t) - \rho_c(h/b)\ddot{k}(t) \end{aligned} \tag{10}$$

Problem II: a time-dependent pressure is applied on the surface of the case; no pressure inside the cylinder:

$$\begin{aligned} &\text{at } r = a \qquad \sigma_r = 0 \\ &\text{at } r = b \qquad \sigma_r = -\sigma_e(t) - (h/b)(E'/b^2)k(t) - \rho_c(h/b)\ddot{k}(t) \end{aligned} \tag{11}$$

Application of the boundary conditions Eqs. (10) and (11) to (6) yields

$$\rho_p M\ddot{k}(t) + \left(\frac{h}{b}\right)\frac{E'}{b^2}k(t) + 2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\int_{0-}^{t} G(t - s)dk(s) = -q(t) \tag{12}$$

where

$$M = \ln(b/a) + (\rho_c/\rho_p)(h/b) \tag{13}$$

and

$$q(t) = -\sigma_i(t) \text{ for Problem I} \tag{14}$$

$$q(t) = +\sigma_e(t) \text{ for Problem II} \tag{15}$$

The solution of the integro-differential equation, Eq. (12), is subject to the initial conditions, Eq. (7). Removing the discontinuity at $t = 0$ and integrating by parts, Eq. (12) is rewritten as

$$\rho_p M\ddot{k}(t) + \left(\frac{G_g}{a^2}\right) Nk(t) + 2\left[\left(\frac{b^2 - a^2}{a^2b^2}\right)\right]\int_{0+}^{t} G'(t-s)k(s) = -q(t) \quad (16)$$

where the prime denotes differentiation with respect to the argument. In Eq. (16)

$$N = 2(b^2 - a^2)/b^2 + (h/b)(E'/G_g)(a^2/b^2) \quad (17)$$

and G_g is the glassy shear modulus.

Elastic Solution

If the material of the encased cylinder is elastic with shear modulus G rather than viscoelastic, Eq. (16) reduces to an ordinary differential equation. For convenience, a dimensionless time η is introduced:

$$\eta = tc/a \qquad \text{where} \qquad c^2 = G/\rho_p \quad (18)$$

We obtain

$$M\ddot{k}(\eta) + Nk(\eta) = -q(\eta)a^2/G \quad (19)$$

The solution of Eq. (19) is

$$k(\eta) = -\left(\frac{a^2}{MG\omega}\right)\int_0^{\eta} q(\eta - s)\sin(\omega s)ds \quad (20)$$

where

$$\omega^2 = N/M \quad (21)$$

Inspection of Eqs. (13, 17, and 21) shows that the parameters for the elastic problem are the structural parameters (h/b) and (b/a), the material parameters (ρ_c/ρ_p) and (E'/G), and the externally applied pressures. In order to investigate the influence of variation of the structural and the material parameters, we assume that the external pressures are suddenly applied and are maintained constant thereafter. The function $k(\eta)$ can now be determined in explicit form. Substitution of $k(\eta)$ into the equation for the radial stress results in

$$\sigma_r = P - (1/M)[\cos\omega\eta \ln(r/a) + 2(1/\omega^2)(1 - a^2/r^2)(1 - \cos\omega\eta)]Q \quad (22)$$

where, for Problem I,

$$P = Q = -\sigma_i H(\eta) \quad (23)$$

and, for Problem II,

$$P = 0 \qquad Q = +\sigma_e H(\eta) \quad (24)$$

The circumferential stress σ_θ is obtained from Eqs. (5) and (22):

$$\sigma_\theta = P - (1/M)[\cos\omega\eta \ln(r/a) + 2(1/\omega^2)(1 + a^2/r^2)(1 - \cos\omega\eta)]Q \quad (25)$$

Of primary interest in problems of the type that are discussed in this paper is the radial bond stress at $r = b$. This radial bond stress is compressive if the inside of the cylinder is pressurized (Problem I). Tensile bond stresses can develop if an external pressure is applied to the case (Problem II), and if the stiffness of the case is sufficiently higher than the shear stiffness of the cylinder material. The latter situation is prevalent for an encased solid propellant, where values of $E'/G = 10^4$ are not uncommon. For Problem II, the radial bond stress at $r = b$ is shown in Fig. 2 for various values of E'/G. The values of the other parameters are taken as $\rho_c/\rho_p = 0.2$, $h/b = 10^{-2}$, and $b/a = 3$. The tensile peaks in the bond stress may cause the propellant to separate from the case. This effect is of course undesirable and it should be prevented by appropriate prestressing.

Viscoelastic Solution

It is again convenient to use a dimensionless time

$$\zeta = tc_R/a \quad \text{where } c_R^2 = G_R/\rho_p \quad (26)$$

In Eq. (26), G_R is the rubbery modulus of the viscoelastic material. Rewriting Eq. (16) in terms of ζ we obtain

$$M\ddot{k}(\zeta) + N^*k(\zeta) + U\int_{0+}^{\zeta} G'(\zeta - s)k(s)ds = \left(\frac{-q(\zeta)a^2}{G_R}\right) \quad (27)$$

where

$$N^* = G_g N/G_R \quad (28)$$

$$U = 2(b^2 - a^2)/b^2G_R \quad (29)$$

Integrodifferential equations of the Eq. (27) type have been discussed by Volterra.[5] By integrating twice with respect to ζ from 0 to ζ, a Volterra integral equation of the second kind is obtained:

$$Mk(\zeta) + \int_0^{\zeta} K(\zeta - s)k(s)ds = -\left(\frac{a^2}{G_R}\right)\int_0^{\zeta} dv \int_0^{v} q(s)ds \quad (30)$$

in which

$$K(s) = \left(\frac{ha^2E'}{b^3G_R}\right)s + U\int_0^{s} G(v)dv \quad (31)$$

The method of solution of Eq. (27) or (30) depends on the complexity of $G(t)$. An analytical solution of Eq. (27) can be obtained by the Laplace transform technique if $G(t)$ is a simple function of t. For practical problems, however, the information on $G(t)$ is usually in the form of experimental

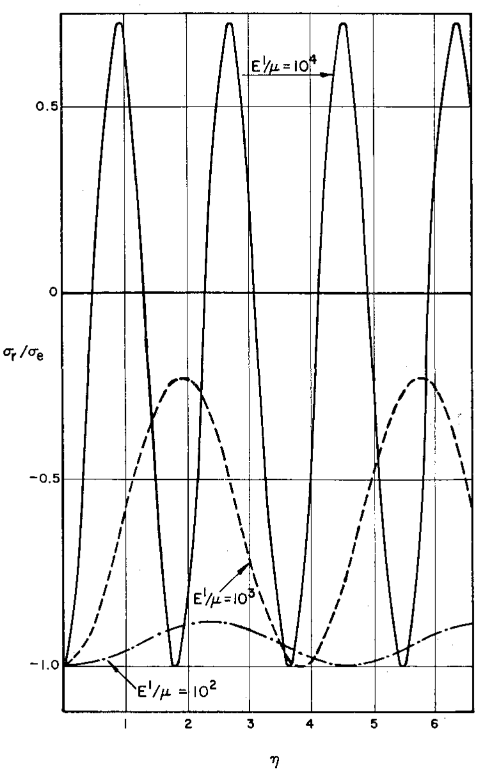

Fig. 2 Problem II: Radial bond stress for an encased elastic cylinder with shear modulus $G = \mu$.

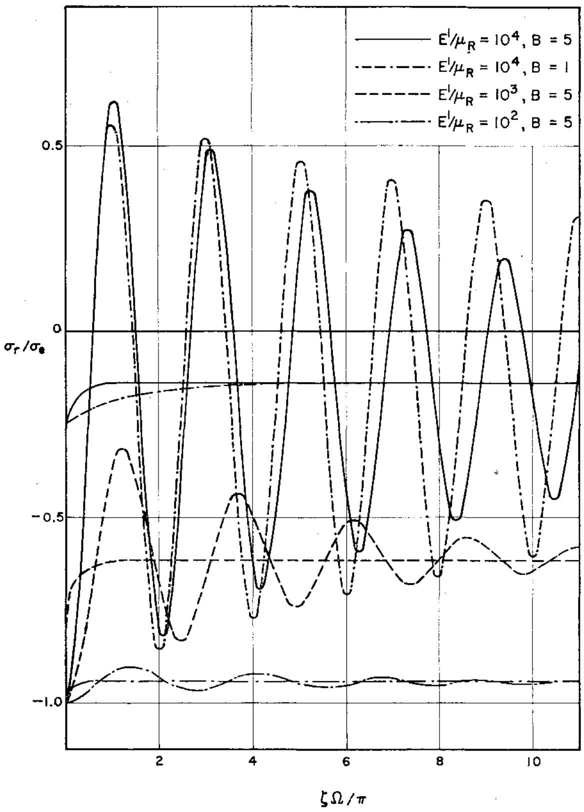

Fig. 3 Problem II: Radial bond stress for an encased viscoelastic cylinder with rubbery shear modulus $G_R = \mu_R$.

results that do not allow a simple functional representation. The function $K(s)$ can, in that case, be determined by a numerical integration of the shear relaxation function vs time curve. A numerical procedure to obtain a solution of Volterra integral equations of the second kind, such as Eq. (30), has been described by Lee and Rogers.[6]

It is not the intention in the present paper to provide a numerical solution for a particular viscoelastic material for which relaxation data are available. The purpose is rather to exhibit the type of solution for a class of viscoelastic materials, namely, materials that show initial and delayed elasticity. An analytical solution and some parametric comparisons are, therefore, provided for an encased cylinder of standard linear viscoelastic solid, which is the simplest model of that class of viscoelastic materials.

Solution of Eq. (27) for the Standard Linear Viscoelastic Solid

If $G(t)$ is a simple function, Eq. (27) can be solved by the Laplace transform method. Application of the Laplace transform over ζ yields

$$\bar{k}(p)[Mp^2 + (ha^2E'/b^3G_R) + Up\bar{G}(p)] = -\bar{q}(p)a^2/G_R \quad (32)$$

The relaxation function in shear is easiest obtained from the well-known differential relation of the standard linear viscoelastic solid

$$s_{ij} + \tau^*\partial s_{ij}/\partial t = 2G_R(e_{ij} + \tau\partial e_{ij}/\partial t) \quad (33)$$

The constants τ^*, G_R, and τ are, respectively, the relaxation time, the rubbery shear modulus, and the retardation time. With the viscoelastic solution for the standard linear viscoelastic solid are always associated two elastic solutions corresponding, respectively, to the initial elastic response (shear modulus $\tau G_R/\tau^*$) and the delayed elastic response (shear modulus G_R). The dynamic viscoelastic solution approaches the static solution for the second elastic case asymptotically.

The dimensionless time ζ is introduced in Eq. (33). After application of the Laplace transform we obtain

$$\bar{s}_{ij} = 2G_R\bar{e}_{ij}(1 + pc_R\tau/a)/(1 + pc_R\tau^*/a) \quad (34)$$

and, thus,

$$p\bar{G}(p) = G_R(1 + pc_R\tau/a)/(1 + pc_R\tau^*/a) \quad (35)$$

After some manipulation, the following expression for $k(p)$ is obtained from Eqs. (35) and (32):

$$M(G_R/a^2)\,\bar{k}(p) = -p\bar{q}(p)(1 + B/p)F(p) \quad (36)$$

where

$$B = a/c_R\tau^* \quad (37)$$

$$F(p) = 1/(p^3 + Bp^2 + Dp + BJ) \quad (38)$$

$$D = [2(\tau/\tau^*)(b^2 - a^2)/b^2 + (h/b)(E'/G_R)(a^2/b^2)]/M \quad (39)$$

$$J = [2(b^2 - a^2)/b^2 + (h/b)(E'/G_R)(a^2/b^2)]/M \quad (40)$$

The two elastic solutions are obtained by substitution of $\tau^* = \infty$, $\tau/\tau^* = \text{const} > 1$, and $\tau^* = \tau = 0$, respectively. It is noted that the two elastic solutions are periodic with frequencies equal to the square roots of D and J, respectively, where $D > J$. It is then to be expected that the viscoelastic solution is also periodic with a frequency in a frequency range which is bounded by the frequencies of the two elastic solutions.

Let the external pressures again be applied as step functions. The inversion of Eq. (36) is then elementary. For the general underdamped problem, $F(p)$ can be written as

$$F(p) = 1/(p - \alpha)(p - \beta - i\gamma)(p - \beta + i\gamma) \quad (41)$$

The inverse Laplace transform of Eq. (41) is

$$(\zeta) = L^{-1}[F(p)] = \frac{e^{\alpha\zeta} - e^{\beta\zeta}[\cos(\gamma\zeta) + (\alpha - \beta)\sin(\gamma\zeta)/\gamma]}{(\alpha - \beta)^2 + \gamma^2} \quad (42)$$

The solution $k(\zeta)$ can now be expressed as

$$M\left(\frac{G_R}{a^2}\right)k(\zeta) = -Q\left[f(\zeta) + B\int_0^\zeta f(s)ds\right] \quad (43)$$

The parameter B is defined by Eq. (37); Q is given by Eqs. (23) and (24).

Employing Eq. (43), the stress in radial direction is found as

$$\sigma_r(r, \zeta) = P - \left(\frac{Q}{M}\right)[f''(\zeta) + Bf'(\zeta)]\ln\left(\frac{r}{a}\right) - \left(\frac{2Q}{M}\right)\left(1 - \frac{a^2}{r^2}\right)\left[B\int_0^\zeta f(s)ds + \left(\frac{\tau}{\tau^*}\right)f(\zeta)\right] \quad (44)$$

where P and Q are defined by Eqs. (23) and (24). The circumferential stress is obtained from Eqs. (44) and (1) as

$$\sigma_\theta(r, \zeta) = P - \left(\frac{Q}{M}\right)[f''(\zeta) + Bf'(\zeta)]\ln\left(\frac{r}{a}\right) - \left(\frac{2Q}{M}\right)\left(1 + \frac{a^2}{r^2}\right)\left[B\int_0^\zeta f(s)ds + \left(\frac{\tau}{\tau^*}\right)f(\zeta)\right] \quad (45)$$

The six independent parameters in the viscoelastic problem are (h/b), (b/a), (ρ_c/ρ_p), $(a/c_R\tau^*)$, (τ/τ^*), and (E'/G_R). A variation of one of the parameters influences the frequency γ and the damping factors α and β, Eq. (42). The frequency γ strongly depends on the magnitudes of D and J, Eqs. (39) and (40), respectively. The material parameters in D and J are, respectively, the ratio of the glassy shear modulus/rubbery shear modulus $= \tau/\tau^*$ and the ratio E'/G_R. For a certain geometry either term may dominate D. In many instances the rigidity of the case is large as compared to the long-time (rubbery) shear stiffness of the propellant. The term $(hE'a^2/G_Rb^3)$ then dominates D and J, and the frequency γ is mainly determined by this term.

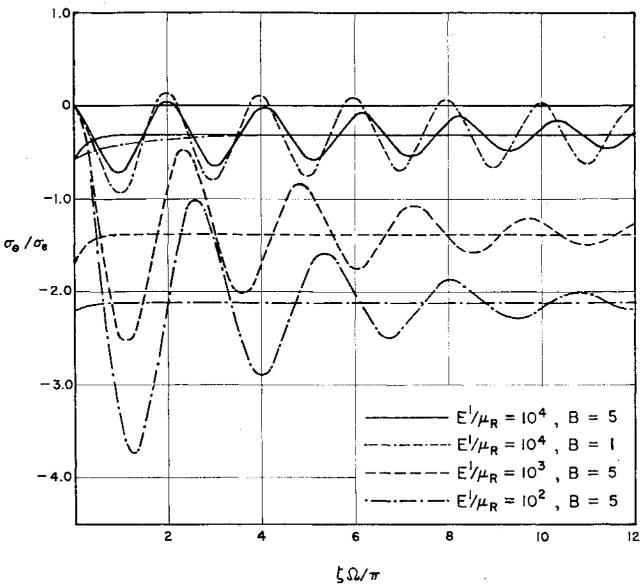

Fig. 4 Problem II: circumferential stress at $r = a$ for an encased viscoelastic cylinder with rubbery shear modulus $G_R = \mu_R$.

Some numerical calculations have been carried out for the following numerical values of the parameters: $(E'/G_R) = 10^4$, 10^3, and 10^2, $(h/b) = 10^{-2}$, $(a/c_R\tau^*) = 5$ and 1, $\tau/\tau^* = 2$, and $(b/a) = 3$. The results are shown in Figs. 3 and 4. The stresses are plotted as functions of $\zeta\Omega/\pi$, where Ω is the dimensionless frequency of the solutions for an encased elastic cylinder with shear modulus $G_R\tau/\tau^*$, thus $\Omega^2 = D$. For comparison, the quasi-static solutions are also shown in Figs. 3 and 4. If a constant external pressure is suddenly applied to the case, the quasi-static solution for the radial stress is found as

$$\frac{\sigma_r}{\sigma_e} = -\frac{2(1 - a^2/r^2)}{MD}\left[\frac{D}{J} - \left(\frac{D}{J} - \frac{\tau}{\tau^*}\right)\exp\left(-\frac{J}{D}B\zeta\right)\right] \tag{46}$$

The quasi-static solution for the circumferential stress is

$$\frac{\sigma_\theta}{\sigma_e} = -\frac{2(1 + a^2/r^2)}{MD}\left[\frac{D}{J} - \left(\frac{D}{J} - \frac{\tau}{\tau^*}\right)\exp\left(-\frac{J}{D}B\zeta\right)\right] \tag{47}$$

where the constants M, B, D, and J are defined by, respectively, Eqs. (13, 37, 39, and 40).

The radial bond stresses are always compressive for Problem I, i.e., if a time-dependent pressure is applied to the inner surface of the cylinder. For large values of E'/G_R, the radial bond stress shows tensile peaks if an external pressure is applied to the case (Fig. 3). Due to the viscoelasticity of the grain, these peaks are damped rapidly and the quasi-static compressive solution is reached. If pressures of equal orders of magnitude are applied to the shell and internally to the cylinder, the radial bond stress can have tensile peaks if the applied pressures are out of phase. The circumferential stress at $r = a$ is shown in Fig. 4. For Problem II, the circumferential stress is generally compressive. It is clear that tensile stresses at the bond between viscoelastic cylinder and elastic shell should be prevented by appropriate prestressing.

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